Hausdorff dimension, Mean quadratic variation of infinite self-similar measures*

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(Received on 10 October, 1996; Accepted on 15 August, 1997)

Abstract: Under weaker condition than that of Riedi & Mandelbrot, the Hausdorff (and Hausdorff-Besicovitch) dimension of infinite self-similar set $K \subset \mathbf{R}^d$ which is the invariant compact set of infinite contractive similarities $\{S_j(x) = \rho_j R_j x + b_j\}_{j\in\mathbf{N}}$ $(0 < \rho_j < 1, b_j \in \mathbf{R}^d, R_j$ orthogonal) satisfying open set condition is obtained. It is proved (under some additional hypotheses) that the β-mean quadratic variation of infinite self-similar measure is of asymptotic property (as $t \longrightarrow 0$).

Key Words: Hausdorff (and Hausdorff-Besicovitch) dimension, infinite self-similar set/measure, mean quadratic variation.

AMS Classification: 28A80, 00A73.

* Project partially supported by the Tianyuan Foundation of China and Postdoctoral station Foundation of the State Education Committee.

1 Introduction

In this paper, we denote $\mathbf{R}^d$ the d-dimensional Euclidean space, $\mathbf{N}$ the set of natural numbers and $\mathbf{Z}$ the set of integer numbers.

For given finite contractive similarities $\{S_j(x) = \rho_j R_j x + b_j\}_{j=1}^m$ of $\mathbf{R}^d$, where $0 < \rho_j < 1, b_j \in \mathbf{R}^d, R_j$ orthogonal, J.E.Hutchinson [1] proved that there exists unique compact set K_1 satisfying

$$K_1 = \cup_{j=1}^m S_j(K_1).$$

K_1 is called *self-similar set*. If there exists an open set O_1 satisfying $S_j(O_1) \subset O_1$ and $S_i(O_1) \cap S_j(O_1) = \emptyset$ $(i \neq j)$, we call that $\{S_j\}_{j=1}^m$ satisfy *open set condition*. We call that they satisfy *strong open set condition* if the sets $S_j(\overline{O})$ are disjoint. Then

Theorem A (Hutchinson) If $\{S_j\}_{j=1}^m$ satisfy open set condition, then the Hausdorff dimension s' of K_1 is the unique solution of the equation $\sum_{j=1}^m \rho_j^{s'} = 1$.

In [1], he also proved that for given probability vector $P = (P_1, P_2, \cdots, P_m)$ satisfying $\sum_{j=1}^m P_j = 1$, there exists unique probability measure μ_1 on $\mathbf{R}^d$ satisfying

$$\mu_1(\cdot) = \sum_{j=1}^m P_j \mu_1(S_j(\cdot))$$

and the support set of μ_1 is K_1. μ_1 is called *self-similar measure* and $\{P_j\}_{j=1}^m$ is called *weights* of μ_1.

Ka-Sing Lau and Jian-rong Wang [2], and R.S.Strichartz [3-7] have done much study on Fourier analysis of self-similar measure. R.S.Strichartz in [3] (or [7]) discussed many fractal measures. If μ is self-similar measure on $\mathbf{R}^d$, Strichartz [4-6] discussed the asymptotic property (as $r \longrightarrow \infty$) of function

$$H(r) = \frac{1}{r^{d-\beta'}} \int_{|x| \le r} |F(x)|^2 dx,$$

where $F(x) = (d\mu)^\wedge$ and β' is defined by $\sum_{j=1}^m \rho_j^{-\beta'} P_j^2 = 1$.

Let μ be a Borel measure on $\mathbf{R}^d$, f be a Borel measurable function, we use μ_f to denote the measure defined by $\mu_f(E) = \int_E f d\mu$ for any Borel set E in $\mathbf{R}^d$.

It is proved in [5] that if $\{S_j\}_{j=1}^m$ satisfies the strong open set condition, then for the self-similar measure μ defined by natural weights (i.e. $P_j = \rho_j^{\beta'}, \quad \beta' = s'$)

$$\frac{1}{r^{d-\beta'}} \int_{|x| \le r} |(\mu_f)^\wedge|^2 dx = q(r) \int |f|^2 d\mu + E(r) \quad \text{for} \quad \forall f \in L^2(d\mu), \qquad (*)$$

where $E(r) \longrightarrow 0$ as $r \longrightarrow +\infty$, and $q(r)$ is a multiplicative periodic function or a positive constant.

Let μ be a σ-finite measure on $\mathbf{R}^d$, for $0 \le \alpha \le d$, let

$$V_\alpha(t;\mu) = \frac{1}{t^{d+\alpha}} \int_{\mathbf{R}^d} |\mu(B_t(x))|^2 dx,$$

where $B_t(x)$ is the ball of radius t, centered at x. We will call $\limsup_{t\longrightarrow 0} V_\alpha(t;\mu)$ the *upper α-mean quadratic variation* (m.q.v.) of μ, and simply call it α-m.q.v. if the limit exists.

If μ is a self-similar measure on $\mathbf{R}^d$, Ka-sing Lau and Jian-rong Wang [2] proved the following two Theorems

Theorem B([2]) Under some additional conditions, we have

$$\lim_{t\longrightarrow 0}[V_{\beta'}(t;\mu) - p(t)] = 0.$$

where $p(t)$ is a multiplicative periodic function or a positive constant and β' is defined as above.

Theorem C ([2]) If the self-similar measure μ defined by natural weights (i.e. $P_j = \rho_j^{\beta'}, \beta' = s'$), under some additional hypotheses

$$\lim_{t\longrightarrow 0}[\frac{1}{t^{d+\beta'}}\int_{\mathbf{R}^d} |\mu_f(B_t(x))|^2 dx - p(t)\int |f|^2 d\mu] = 0 \quad \text{for} \quad \forall f \in L^2(d\mu),$$

where $p(t)$ is the function in Theorem B.

R.H.Riedi and B.B.Mandelbrot [8] introduced infinite self-similar sets and infinite self-similar measures on $\mathbf{R}^d$ (definitions see later of this paper), discussed multifractal formalism for infinite self-similar measures and the Hausdorff dimension of infinite self-similar sets (under some additional conditions). In this paper, under weaker condition than that of Riedi & Mandelbrot, we extend Theorem A to the infinite self-similar case. If μ is infinite self-similar measure and the equation $\sum_{j=1}^\infty P_j^2\rho_j^{-\beta} = 1$ has finite solution β, then under some additional hypotheses, R.S.Strichartz [5] obtained the asymptotic property of function $H(r)$ and conclusion (*). In this paper, we also extend Theorem B,C to the infinite self-similar case.

2 Hausdorff (and Hausdorff-Besicovitch) dimension of infinite self-similar set.

For given infinite contractive similarities $\{S_j(x) = \rho_j R_j x + b_j\}_{j\in\mathbf{N}}$ of $\mathbf{R}^d$, where $0 < \rho_j < 1, b_j \in \mathbf{R}^d, R_j$ orthogonal, from [8], there exists unique compact set K satisfying

$$K = \overline{\cup_{j=1}^\infty S_j(K)}.$$

K is called *infinite self-similar set*. K can be constructed as following. Let $E_0 \subset \mathbf{R}^d$ be a compact set, denote $E_{j_1\cdots j_k} = S_{j_1}\circ\cdots\circ S_{j_k}(E_0)$, then

$$K = \cap_{k=0}^\infty \overline{\cup_{j_1,\cdots,j_k\in\mathbf{N}} E_{j_1\cdots j_k}}.$$

For given probability sequence $(P_1, P_2, \cdots)$ with $\sum_{j=1}^\infty P_j = 1$, from [8], there exists unique probability measure μ on $\mathbf{R}^d$ satisfying

$$\mu(\cdot) = \sum_{j=1}^\infty P_j \mu(S_j(\cdot)).$$

We call μ *infinite self-similar measure* and $\{P_j\}_{j=1}^{\infty}$ *weights* of μ. Its support set is K.

Definition 1 *We call $\{S_j(x)\}_{j\in\mathbf{N}}$ satisfying open set condition if there exists a bounded open set $O \subset \mathbf{R}^d$ such that $S_j(O) \subset O$ and $S_i(O) \cap S_j(O) = \emptyset$ $(i \neq j)$.*

For any subset $A \in \mathbf{R}^d$ and $0 \leq s < \infty$, let $\mathcal{M}_\delta^s(A) = \inf \sum_{i=1}^{\infty} |A_i|^s$, where $A = \cup_{i=1}^{\infty} A_i$ is a countable decomposition of A into subsets of diameter $|A_i| < \delta$ (> 0). We set $|A_i|^0 = 0$ if A_i is empty and $|A_i|^0 = 1$ otherwise. The the s-dimensional measure of A is defined to be

$$\mathcal{M}^s(A) = \sup_{\delta > 0} \mathcal{M}_\delta^s(A).$$

The Hausdorff-Besicovitch dimension[9] of A is

$$\dim_M(A) = \sup\{0 \leq s < \infty : \quad \mathcal{M}^s(A) > 0\}.$$

Remark: It is easy to see that in the definition of $\mathcal{M}_\delta^s(A)$, we can replace $|A_i|$ by $|\overline{A_i}|$.
From the definition of fractal dimension[10] $\dim_H(A)$, we can see that

$$\dim_H(A) \leq \dim_M(A). \tag{1}$$

Theorem 1 *If the equation $\sum_{j=1}^{\infty} \rho_j^s = 1$ has finite solution s, and $\{S_j\}_{j=1}^{\infty}$ satisfy open set condition, K is the infinite self-similar set, then the Hausdorff-Besicovitch dimension $\dim_M(K)$ and Hausdorff dimension $\dim_H(K)$ of K is s.*

Remark. Our condition is weaker than Riedi & Mandelbrot's [8] condition: there exist numbers r, R such that $-\infty < \log r \leq (1/j) \log \rho_j \leq \log R < 0$ $\forall j$.

Proof of Theorem 1 To get the upper bound. Let $K = \cup_{i=1}^{\infty} A_i$ be any decomposition of K into subsets of diameter $< \delta$, then a new decomposition is provided by $K = \cup_{i=1}^{\infty} \overline{\cup_{j=1}^{\infty} A_{ij}}$, where $A_{ij} = \varphi_j(A_i)$. Because

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |A_{ij}|^s &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\rho_j|^s |A_i|^s \\ &\leq (\sum_{j=1}^{\infty} \rho_j^s) \sum_{i=1}^{\infty} |A_i|^s, \end{aligned}$$

it follows that whenever $\sum_{j=1}^{\infty} \rho_j^s < 1$ we must have $\mathcal{M}_\delta^s(K) = 0$, then $\mathcal{M}^s(K) = 0$. As $\dim_M(K) = \inf\{s : \quad \mathcal{M}^s(K) = 0\}$, hence $\dim_M(K) \leq s$ where $\sum_{j=1}^{\infty} \rho_j^s = 1$. From (1), we have $\dim_H(K) \leq s$.

To get the lower bound. We let $K^{(m)}$ be the self-similar set generated by $\{S_j\}_{j=1}^{m}$, then from Theorem 8 of ref.[11], we have

$$\dim_M(K^{(m)}) \geq \min\{d, s^{(m)}\}, \tag{2}$$

where $s^{(m)}$ is the positive solution of $\sum_{j=1}^{m} \rho_j^{s^{(m)}} = 1$. Using Theorem 4.13 of ref.[10], similar to the proof of Theorem 8 of ref.[11], we can obtain

$$\dim_H(K^{(m)}) \geq \min\{d, s^{(m)}\}. \tag{3}$$

Then from Lemma 8 of ref.[8], we have $\lim_{m\longrightarrow\infty} s^{(m)} = s$, where $\sum_{j=1}^{\infty} \rho_j^s = 1$. Since for any m, $K^{(m)} \subset K$, we have $\dim_M(K) \geq \dim_M(K^{(m)})$ and $\dim_H(K) \geq \dim_H(K^{(m)})$. From open set condition, we have $s < d$, then from (2) and (3), we have

$$\dim_M(K) \geq s^{(m)} \tag{4}$$

and

$$\dim_H(K) \geq s^{(m)}. \tag{5}$$

Take limit from (4) and (5), we have $\dim_M(K) \geq s$ and $\dim_H(K) \geq s$. #

The method used in proof of Theorem 1 can be used to estimate the Hausdorff (and Hausdorff-Besicovitch) dimension of the limit set of infinite non-similar contractive maps.

Corollary 1 *Let $\{\varphi_j\}_{j=1}^{\infty}$ be infinite contractive maps with*

$$|\varphi_j(x) - \varphi_j(y)| \leq c_j|x - y|, \quad x, y \in \mathbf{R}^d, \quad j = 1, 2, \cdots,$$

and satisfying open set condition, and denote E their contractive-invariant set. If the equation $\sum_{j=1}^{\infty} c_j^u = 1$ has finite solution u, then $\dim_H(E) \leq \dim_M(E) \leq u$.

Corollary 2 *Let $\{\varphi_j\}_{j=1}^{\infty}$ be infinite contractive maps with*

$$|\varphi_j(x) - \varphi_j(y)| \geq b_j|x - y|, \quad x, y \in \mathbf{R}^d, \quad j = 1, 2, \cdots,$$

and satisfying open set condition, and denote E their contractive-invariant set. If the equation $\sum_{j=1}^{\infty} b_j^l = 1$ has finite solution l, then $\dim_M(E) \geq \dim_H(E) \geq \min\{d, l\}$.

Proof. Since $\{\varphi_j\}$ are non-similar maps, we can not obtain $l^{(m)} \leq d$ from open set condition, where $l^{(m)}$ satisfies $\sum_{j=1}^{m} b_j^{l^{(m)}} = 1$. then similar to proof of Theorem 1, this conclusion holds. #

3 Mean quadratic variations of infinite self-similar measures.

We define

$$H(r) = \frac{1}{r^{d-\beta}} \int_{|x|\leq r} |F(x)|^2 dx,$$

where $F(x)$ is the Fourier transform of μ.

If μ is a Borel measure on $\mathbf{R}^d$, for every μ-measurable function f, we use μ_f to denote the measure $\mu_f(E) = \int_E f d\mu$ for any Borel subset E.

Definition 2 . *If in addition to the definition of open set condition, the sets $S_j(\overline{O})$ are mutually disjoint and O intersects K, we call $\{S_j\}_{j\in\mathbf{N}}$ satisfy strong open set condition.*

We assume $\{S_j\}_{j=1}^{\infty}$ satisfy strong open set condition. Let d_{jk} denote the distance between $S_j(O)$ and $S_k(O)$ which is positive for $j \neq k$ by strong open set condition. We assume

$$\sum_{j\neq k} P_j P_k d_{jk}^{-\beta} < \infty. \tag{6}$$

Denote $q(\lambda) = \sum_{\rho_j \le \lambda} P_j^2 \rho_j^{-\beta}$, we assume

$$q(\varepsilon\lambda) \le \delta q(\lambda) \tag{7}$$

for some $0 < \varepsilon < 1$ and $0 < \delta < 1$.

Under the conditions (6) and (7), R.S.Strichartz [5] (P357-P358) obtained the asymptotic property (as $r \longrightarrow +\infty$) of the function $H(r)$ and conclusion (*) for infinite self-similar measures.

We use $J = (j_1, j_2, \cdots, j_k)$ to denote the multi-index, $|J| = k$ its length, and Λ the set of all such multi-indice, where $j_i \in \mathbf{N}, \quad i = 1, \cdots, k$ and $k \in \mathbf{N}$. We set

$$P_J = P_{j_1} P_{j_2} \cdots P_{j_k}, \qquad \rho_J = \rho_{j_1} \cdots \rho_{j_k}, \qquad E_J = E_{j_1 j_2 \cdots j_k}$$

For any $0 < t < 1$, we denote

$$\Lambda(t) = \{J \in \Lambda : \quad \rho_J = \sup \rho_{J'}, \quad \rho_{J'} < t\},$$

and for fixed parameter ε (given in condition (7)), we denote

$$\Lambda_1(t) = \{J \in \Lambda(t) : \quad \rho_J \ge \varepsilon t\}.$$

Then we have

Theorem 2 *Let μ be infinite self-similar measure, we assume that the condition (7) holds, then $V_\beta(t;\mu)$ is bounded below by a positive constant on $0 < t \le 1$.*

Proof. Since $\sum_{j=1}^{\infty} P_j^2 \rho_j^{-\beta} = 1$, then $\sum_{J\in\Lambda(t)} P_J^2 \rho_J^{-\beta} = 1$. When $J \in \Lambda_1(t)$, we have $\varepsilon t \le \rho_J < t$. Hence

$$t^{-\beta} < \rho_J^{-\beta} \le (\varepsilon t)^{-\beta}.$$

From the condition (7) and similar to ref.[5](P358), we can prove

$$\sum_{J\in\Lambda_1(t)} P_J^2 \rho_J^{-\beta} \ge (\delta^{-1} - 1) \sum_{J\in\Lambda(t)} P_J^2 \rho_J^{-\beta}.$$

Hence

$$\begin{aligned}
(\delta^{-1} - 1) &= (\delta^{-1} - 1) \sum_{J\in\Lambda(t)} P_J^2 \rho_J^{-\beta} \le \sum_{J\in\Lambda_1(t)} P_J^2 \rho_J^{-\beta} \\
&\le \sum_{J\in\Lambda_1(t)} P_J^2 (\varepsilon t)^{-\beta} \le \sum_{J\in\Lambda(t)} P_J^2 (\varepsilon t)^{-\beta},
\end{aligned}$$

hence

$$\frac{1}{t^\beta}\sum_{J\in\Lambda(t)} P_J^2 \geq (\delta^{-1}-1)\varepsilon^\beta.$$

Without loss of generality we assume $|E_0| = 1$. We denote ω_d the Lebesgue measure. Note that μ is supported by $\cup\{E_J : \quad J\in\Lambda(t)\}$ and $\mu(E_J) = P_J$. Hence

$$\begin{aligned} V_\beta(t;\mu) &= \frac{1}{t^{d+\beta}}\int[\int\int \chi_{B_t(x)}(\xi)\chi_{B_t(x)}(\eta)d\mu(\xi)d\mu(\eta)]dx \\ &= \frac{1}{t^{d+\beta}}\int\int \omega_d(B_t(\xi)\cap B_t(\eta))d\mu(\xi)d\mu(\eta) \\ &\geq \frac{1}{t^{d+\beta}}\sum_{J\in\Lambda(t)}\int\int_{\xi,\eta\in E_J} \omega_d(B_t(\xi)\cap B_t(\eta))d\mu(\xi)d\mu(\eta). \end{aligned}$$

Since $|E_J| = \rho_J \leq t$, hence $B_t(\xi)\cap B_t(\eta)$ contains a ball of radius $t/2$ whenever $\xi,\eta\in E_J$. It follows that

$$\begin{aligned} V_\beta(t;\mu) &\geq \frac{c}{t^\beta}\sum_{J\in\Lambda(t)}\int\int_{\xi,\eta\in E_J} d\mu(\xi)d\mu(\eta) \\ &\geq \frac{c}{t^\beta}\sum_{J\in\Lambda(t)} P_J^2 \geq c(\delta^{-1}-1)\varepsilon^\beta, \end{aligned}$$

where $c(\delta^{-1}-1)\varepsilon^\beta$ is a positive constant. #

From the asymptotic property of $H(r)$ of infinite self-similar measure ([5]), Theorem 4.10 and Corollary 4.12 of [2] and our Theorem 2, we have

Theorem 3 *Let μ be infinite self-similar measure. Assume conditions (6) and (7) hold, then*

$$\lim_{t\longrightarrow 0}(V_\beta(t;\mu) - P(t)) = 0$$

for some $P > 0$ such that the following holds.

(i) If $\{-\ln\rho_j : \quad j\in\mathbf{N}\}$ is non-arithematic, then $P(t) = c'$ for some constant c'.

(ii) Otherwise, let $((\ln\rho)\mathbf{Z}), \quad \rho > 1$ be the lattice generated by $\{-\ln\rho_j : \quad j\in\mathbf{N}\}$, then $P(\rho t) = P(t)$.

From the conclusion (*) of infinite self-similar measure ([5]), Theorem 4.10 and Corollary 4.12 of [2], if the equation $\sum_{j=1}^\infty \rho_j^s = 1$ has finite solution s, then

Theorem 4 *Let μ be infinite self-similar measure with natural weights $P_j = \rho_j^\beta$, where $\beta = s$ is the finite solution of equation $\sum_{j=1}^\infty \rho_j^s = 1$, we assume conditions (6) and (7) holds, then for any $f\in L^2(d\mu)$ we have*

$$\lim_{t\longrightarrow 0}[\frac{1}{t^{d+\beta}}\int|\mu_f(B_t(x))|^2dx - P(t)\int|f|^2d\mu] = 0,$$

where P defined in Theorem 3.

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